A Diagrammatic Proof That Indirect Utility Functions Are Quasi-Convex

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Duality theory has become part of the standard training of economics graduate students and many upper-level undergraduates. My teaching experience suggests that students have little problem understanding why the cost function is concave or why the indirect profit function is convex, because a simple envelope-type diagrammatic argument is readily available (e.g., Kreps 1990; Silberberg 1990; Varian 1984). Many students, however, have difficulty grasping the quasi-convexity of the indirect utility function. Although an algebraic proof can be found in textbooks such as Kreps (1990) and Varian (1984), those who are uninitiated in mathematical language may find the proof unilluminating if not unintelligible. I have constructed a diagrammatic proof for use in class, and students seem to be quite receptive. As a bonus, the proof leads to Roy's identity. The student should be reminded, however, that the diagrammatic proof is meant to supplement—not to replace—the algebraic treatment.

The indirect utility function, \( v(p_1, p_2, y) \), gives the maximum achievable utility when prices are \( p_1, p_2 \) (in the two-good case) and when money income is \( y \). A convenient way to represent an indirect utility function is the price-indifference diagram. A price-indifference curve shows combinations of \( p_1 \) and \( p_2 \) such that the consumer is indifferent. It is downward sloping because, when \( p_1 \) is increased, the consumer can maintain the same utility level only if \( p_2 \) is reduced. Unlike ordinary indifference curves, however, the direction of preference points to the southwest: lower prices are preferred to higher prices. This is a reflection of the fact that indirect utility is a decreasing function in prices.

A function \( v(p_1, p_2, y) \) is quasi-convex in \( (p_1, p_2) \) if and only if its lower contour set—the set of prices \( (p_1, p_2) \) such that \( v(p_1, p_2, y) \leq k \)—is convex for any \( k \). Because combinations of prices to the northeast of a price-indifference curve yield lower utility than those on the curve, they constitute a lower contour set for the indirect utility function. To prove that the indirect utility function is quasi-convex, it is thus necessary and sufficient to show that the price-indifference curves are convex to the origin. (When a utility function is quasi-concave, the associated indifference curves are also convex to the origin. The apparent inconsistency in terminology is resolved if one notices that ordinary indifference diagrams and price-indifference diagrams have opposite preference directions.)
In Figure 1, the line $VV'$ shows a price-indifference curve. Any point in
the shaded region is preferred to any point on line $VV'$. As drawn in Figure
1, the price-indifference curve is convex to the origin. To see why this must
be the case, consider an arbitrary point $E = (p_0^0, p_0^1)$. Let $(x_1^0, x_2^0)$ be the op-
timal consumption bundle associated with that set of prices and income.
Draw a line $AA'$ passing through $E$ with slope $-x_1^0/x_2^0$. Any combination of
prices on line $AA'$ has the property that the bundle $(x_1^0, x_2^0)$ is affordable with
the initial budget. (More precisely, the equation for line $AA'$ is $p_1 x_1^0 + p_2 x_2^0
= p_1^0 x_1^0 + p_2^0 x_2^0$. The slope of the line is therefore $dp_2/dp_1 = -x_1^0/x_2^0$, as
stated.) Thus, if indirect utility at $E$ is $v_0$, the level of utility at any point on
$AA'$ must be at least $v_0$. When prices change from point $E$ to some other
point on $AA'$, the consumer can maintain the same utility level by keeping
his consumption bundle unchanged. Usually he can do better by substitut-
ing away from the good that is now more expensive. It follows that the
price-indifference curve for utility level $v_0$ must lie within area $AEC$ to the
left of $E$ and within $A'EC'$ to the right of $E$. $VV'$ is therefore bounded be-
low by $AA'$ and is tangent to $AA'$ at $E$. (The tangency condition obtains
when the consumer maximizes utility at a regular interior solution. The stu-
dent may be invited to guess what the price-indifference curve would look
like when $x_1$ and $x_2$ are perfect substitutes or perfect complements.) This in
turn implies that the price indifference curve is convex in the neighborhood
of $E$. Because the point $E$ is arbitrary, the same argument can be used to
show that price-indifference curves are everywhere convex to the origin. In
other words, the indirect utility function is quasi-convex.

![Figure 1: A Convex Price-Indifference Curve](image-url)
Our argument also establishes that, for the regular case, the price-indifference curve has the same slope as $AA'$ at point $E$. Thus, $-\frac{\delta v/\delta p_1}{\delta v/\delta p_2} = -x_i/x_j$. As an exercise, the student can be asked to show that this tangency condition is just a restatement of Roy's identity. (Hint: Apply Euler's theorem to the indirect utility function. Then plug in the tangency condition and use the budget constraint.)

REFERENCES


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